MATH 2210 Name: ____________________

Workshop 5: Group Representations ID: ____________________

Work in your group to complete the following exercises. You may print this handout, annotate the PDF or write your answer on paper. Make your grader's life easier by writing neatly and legibly!

Please include full explanations and write your answers using complete sentences (not just a bunch of mathematical symbols!). It is important to be able to explain your reasoning to someone else in writing.

Warmup

Question 1. Recall that the complex numbers $\mathbb{C}$ are values of the form $a + bi$, with $a, b \in \mathbb{R}$. Show that $\mathbb{C}$ forms a vector space over $\mathbb{R}$.

(a) How do we define addition and scalar multiplication (that is, multiplication by a real number) on the complex numbers?

(b) Verify that these operations satisfy the following two axioms of a vector space (**optional:** check the remaining axioms as well).

$$S2. \quad a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w} \text{ for all } \mathbf{v}, \mathbf{w} \in V \text{ and } a \in \mathbb{R}$$
$$S3. \quad (a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v} \text{ for all } \mathbf{v} \in V \text{ and } a, b \in \mathbb{R}$$

(c) What is the dimension of $\mathbb{C}$ over $\mathbb{R}$? Give two examples of bases for this vector space.

In this worksheet, we'll explore some basics of **representation theory**, a branch of mathematics that studies abstract algebraic structures (such as groups and associative algebras, which we define below) by viewing their elements as linear transformations over a vector space (that is, matrices).

In general, an algebraic structure consists of a (non-empty) collection of elements along with operations over these elements that obey some desired properties. One example of an algebraic structure is a group.

Definition. A **group** $(G, \diamond)$ is a non-empty set G with a well-defined binary operation $\diamond \colon G \times G \to G$ satisfying:

(i) $(g \diamond h) \diamond j = g \diamond (h \diamond j)$ for all $g, h, j \in G$ (associativity),

(ii) there is an element $e \in G$ such that $e \diamond g = g \diamond e = g$ for all $g \in G$ (identity),

(iii) for every $g \in G$ there exits an element $h \in G$ such that $g \diamond h = h \diamond g = e$ (inverse).

The group is called **abelian** (or commutative) if it additionally satisfies:

(iv) $g \diamond h = h \diamond g$ for all $g, h \in G$ (commutativity).

Question 2.

(a) Show that 2×2 matrices form a group under matrix addition by

A. Explaining why this operation is associative (a 1 sentence explanation is enough here).

B. Writing down the identity element and showing it satisfies property (ii).

C. Writing down the inverse of an arbitrary element and showing it satisfies property (iii).

(b) Is this group abelian? If so, explain why. If not, give a counterexample.

Question 3.

(a) Explain why the 2×2 matrices **do not** form a group under matrix multiplication. Which property is violated?

(b) By restricting to a subset of the 2×2 matrices with a special property, we can obtain a group under matrix multiplication. Which subset? Verify that this subset is a group by following the steps of Question 2(a).

(c) Is your group from part (b) abelian? If so, explain why. If not, give a counterexample.

Note that the results from the previous two exercises hold more generally for any $n \times n$ matrices. Perhaps a more surprising result is most groups (including all finite groups) can be represented as a collection of matrices.

Definition. A **representation** of a group $(G, \diamond)$ is a mapping $\varphi : G \to M$, where M is a subset of $n \times n$ matrices for some n, such that:

(i) φ is a bijection. That is, for each matrix $m \in M$, there is a unique $g \in G$ with $\varphi(g) = m$.

(ii) φ preserves the group operation. That is, $\varphi(g \diamond h) = \varphi(g) \cdot \varphi(h)$, where $\cdot$ depicts matrix multiplication.

So a representation allows us to view any group as a collection of linear transformations on a vector space (matrices).

Question 4.

(a) Consider the matrices $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ for some $a, b \in \mathbb{R}$. Evaluate the matrix products:

$AB =$ $\qquad\qquad$ $BA =$

(b) Explain why the map $\varphi : \mathbb{R} \to M$ where $M = \left\{ \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} : r \in \mathbb{R} \right\}$ given by $\varphi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is a representation of $(\mathbb{R}, +)$, the group of real numbers under the addition operation.

Question 5. The Symmetric Groups

One popular collection of groups is the symmetric groups. The symmetric group of order n, S_n contains all of the permutations (reorderings) of a set of n elements. We'll focus on S_3. Two elements of S_3 are,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Here, we've represented the permutations in 2-line notation. The top line is the permutation input and the bottom line is the output (so σ sends 1 to 2, 2 to 3, and 3 to 1).

(a) How many permutations are in S_3? Write them all out.

(b) The operation on S_3 is composition. That is, the product of permutations $\tau \circ \sigma$ is the permutation obtained by first applying σ and then applying τ to the result. Compute the following permutations:

$\tau \circ \sigma =$ $\sigma \circ \tau =$

S_3 is a group under composition. Associativity follows from the associativity of function composition.

(c) What is the identity element of S_3?

(d) Describe, in words, the inverse of a permutation.

(e) Is S_3 abelian? If so, explain why. If not, give a counterexample.

Question 6. Representation of S_3

S_3 is a finite group, so we can find a representation for it. We'll specifically find a representation, φ, with 3×3 matrices, fittingly called the permutation matrices.

(a) Find a 3×3 matrix that realizes the permutation σ (from the previous page).

Hint: Consider multiplying a vector $v = \begin{bmatrix} a & b & c \end{bmatrix}^\intercal$ by this matrix, M. The top entry of Mv is the dot product of the top row of M with v. According to σ, this entry should be the third entry of v, c. This gives us the coefficients of the top for of M. Repeat with the remaining rows.

(b) Find the matrix representations of the remaining permutations in S_3.

(c) Compute the matrix products $\varphi(\tau)\varphi(\sigma)$ and $\varphi(\sigma)\varphi(\tau)$. Compare with your answer to Question 5b.

(d) Describe how to obtain the $n \times n$ matrix representation of an arbitrary permutation in S_n. Feel free to include examples to clarify your explanation.