MATH 2210 Name: ________________________

Workshop 6: An Algebraic Integration Technique ID: ________________________

Work in your group to complete the following exercises. You may print this handout, annotate the PDF or write your answer on paper. Make your grader's life easier by writing neatly and legibly!

Please include full explanations and write your answers using complete sentences (not just a bunch of mathematical symbols!). It is important to be able to explain your reasoning to someone else in writing.

Warmup: A Review from Calculus

Question 1. Compute the following integral.

$$\int e^x \cos(2x)\, dx$$

Hint: This can be done with two applications of integration by parts:

$$\int u(x)v'(x)\, dx = u(x)v(x) - \int v(x)u'(x)\, dx.$$

In this workshop, we'll explore some connections between calculus and linear algebra, and how we can leverage these to compute some complicated integrals.

Question 2. We consider the collection $\mathcal{C}$ of all continuous functions of the real numbers. $\mathcal{C}$ contains many of the functions you're familiar with, such as polynomial functions like $x^2 + 3$, some trig functions like $\sin(x)$ and $\cos(x)$, exponential functions, as well as countless other functions.

Here, we'll argue that $\mathcal{C}$ is a vector space.

(a) How can we define addition and scalar multiplication in $\mathcal{C}$?

(b) What is the 0-vector in $\mathcal{C}$?

(c) Verify that your choices satisfy the axioms of a vector space.

Hint: You'll need to cite some facts about continuous functions that you learned in calculus.

In fact, $\mathcal{C}$ is different from the other vector spaces that we have studied, because it is *infinite dimensional*. This means that given any finite collection of continuous functions $f_1, \ldots, f_n \in \mathcal{C}$, there is a function $f \in \mathcal{C}$ that is not in $\operatorname{span}(f_1, \ldots, f_n)$. For the rest of the worksheet, we'll work only with finite-dimensional subspaces of $\mathcal{C}$.

Question 3.

(a) Explain why $V = \operatorname{span}(e^x, xe^x, x^2e^x)$ is a 3-dimensional subspace of $\mathcal{C}$.

Hint: Set a linear combination of these functions equal to the 0-vector in $\mathcal{C}$, and evaluate both functions at multiple points to solve for the coefficients.

(b) Explain why the derivative acts as a linear transformation from V to itself.

Hint: The derivative is a linear operator (it preserves addition and scalar multiplication), so this amounts to arguing that the derivative maps a basis for V back into V (why?).

(c) Represent the derivative linear transformation as a 3×3 matrix D (acting on coordinate vectors with respect to some basis of V).

Note that D is an invertible matrix. The Fundamental Theorem of Calculus tells us that integration is the inverse operation to differentiation. Therefore, D^{-1} gives the integration map in V [1].

(d) Compute D^{-1}.

(e) Use part (c) to compute the integral,

$$\int (3x^2e^x - 2xe^x + 5e^x)\, dx$$

[1]Since, D^{-1} is a linear function, it computes the antiderivative with constant of integration $C = 0$.

Question 4. As a final exercise, we return to a more general version of the integral from Question 1:

$$\int e^{ax} \cos(bx)\, dx$$

where $a, b \in \mathbb{R}$.

(a) Explain why $W = \operatorname{span}(e^{ax}\cos(bx), e^{ax}\sin(bx))$ is a 2-dimensional subspace of $\mathcal{C}$.

(b) The derivative is a linear transformation from W to itself.

Find a matrix, E, representing this transformation.

(c) Express $e^{ax}\cos(bx)$ as a coordinate vector with respect to the basis W.

(d) Find E^{-1} and use this to calculate an expression for

$$\int e^{ax} \cos(bx)\, dx.$$

(e) Verify that your answer from part (c) agrees with your answer to Question 1.