MATH 2210 Name: _____

Workshop 2: The Area of Polygons ID: _____

Work in your group to complete the following exercises. You may print this handout, annotate the PDF or write your answer on paper. Make your grader's life easier by writing neatly and legibly!

Please include full explanations and write your answers using complete sentences (not just a bunch of mathematical symbols!). It is important to be able to explain your reasoning to someone else in writing.

Warmup

Question 1.

(a) Write out the cofactor expansion for the following matrix determinant, expanding along the last column.

$$\begin{vmatrix} 2 & 1 & 1 \\ 6 & 3 & 1 \\ 3 & 6 & 1 \end{vmatrix}$$

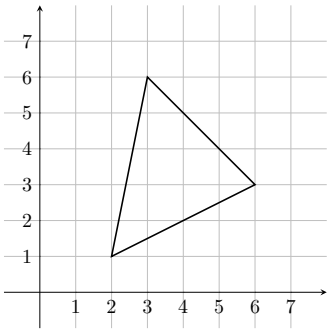
Use this to compute the determinant of the matrix.

(b) Generalize this idea to find the cofactor expansion for the following matrix, which includes variables in some of the entries. Again, expand along the last column to obtain a polynomial expression in these variables.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Question 2. In this question, we'll derive a formula for the area of a triangle involving a determinant. Later, we'll use this to find a formula for the area of any polygon.

(a) Find the area of the triangle below.



Hint: You can express this as the area of a rectangle minus the areas of three right triangles.

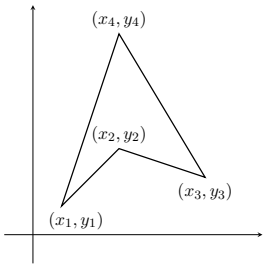
(b) Carry out the same area more generally, replacing the coordinates of the points: $(2, 2) = (x_1, y_1)$, $(6, 3) = (x_2, y_2)$, $(3, 6) = (x_3, y_3)$. Simplify this expression to a polynomial in these variables.

(c) Compare your answers to Question 1(b) and Question 2(b). Use this to write a formula for the area of a triangle involving a determinant.

Note: This formula holds for any triangle, regardless of the positions of the vertices, as long as they are listed in counter-clockwise order.

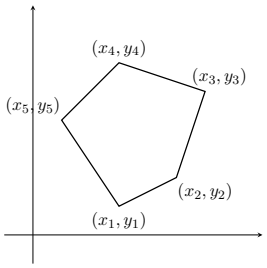
Question 3. We'll extend our formula for the area of a triangle to general polygons.

(a) Find the area of the given quadrilateral, in terms of the coordinate variables.



Hint: Split the quadrilateral into two triangles by drawing the line connecting (x_2, y_2) and (x_4, y_4). Leave your answer in terms of the 2×2 determinants.

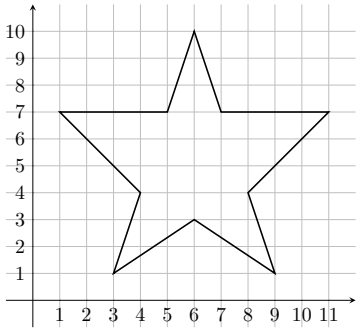
(b) Find a formula for the area of the given pentagon, in terms of the coordinate variables.



Hint: Split the pentagon into three triangles by connecting its vertices. You will end up with the same formula regardless of how you split it. (Optional: think about why this is.)

(c) What is the formula for the area of the polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$?

Question 4. Use the formula that you derived to compute the area of this star.



Question 5. How will your computed area change if you instead traverse the coordinates in a clockwise direction around the polygon?

Question 6 (**Bonus**)**.** The formulas that we derived extend to higher dimensions (allowing us to compute the volume of higher dimensional *polytopes*). Write down a formula for the volume of a tetrahedron (triangular pyramid) in terms of its vertex coordinates. (**Hint:** The volume of a tetrahedron with vertices $(0,0,0),(a,0,0),(0,b,0),(0,0,c)$ is $\frac{abc}{6}$)

Use your formula to compute the volume of the tetrahedron with vertices $(1,3,2),(7,3,6),(8,9,4)$ and $(2,1,5)$.