Recitation 21

1. In this question, we consider a new type of bifurcation point, the *pitchfork bifurcation*. This type of bifurcation is not very common in biological models, so we instead look at a scenario of opinion modeling (for more information, see page 165 of the textbook).

Consider a group of people, each of whom holds either a positive (P) or negative (N) opinion about a specific question (e.g, will the New York Knicks win the NBA championship this year? Will the Dow Jones go up in the coming month?)

(a) The size of the group remains fixed, with only the split between the opinions changing (that is, $P + N = m$ for some constant m). Therefore, we can fully capture the state of the system with a single state variable X, measuring the *tilt towards positive opinion*, and given by the equation

$$X := \frac{P - N}{P + N}.$$

Determine the state space of X.

(b) Provide an interpretation of $X = 0$.

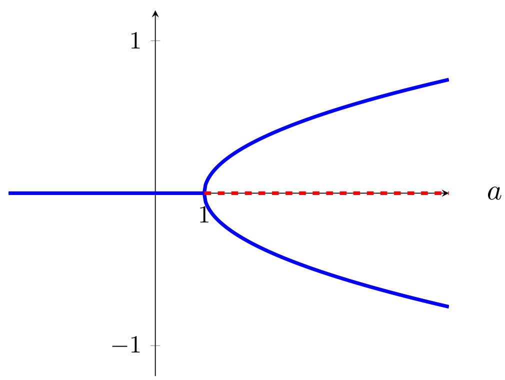
We assume that there is a *bandwagon* effect that influences the opinions. This means that people are more likely to switch to an opinion that is held by more people. By letting the free parameter a represent the strength of this bandwagon effect, we can represent our opinion modeling system (again, the details are in the textbook) by the change equation

$$X' = (1 - X)e^{aX} - (1 + X)e^{-aX}$$

(c) Determine the equilibria and sketch a phase portrait when $a = 0$.

(d) Verify that $\frac{1}{2}, -\frac{1}{2}$ and 0 are equilibria when $a = \ln(3) \approx 1.098$. Use this to sketch the phase portrait.

The bifurcation diagram for this system is

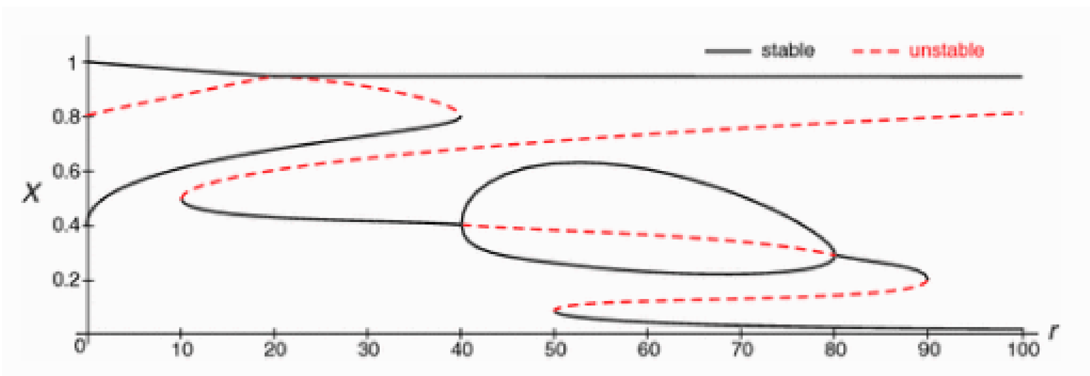


(e) Suppose $a < 1$. Describe the long-term behavior of the system.

(f) Suppose $a > 1$. Describe the long-term behavior of the system.

(g) Suppose that a is large and that there is a perfect split in the opinion. What happens if there is a tiny fluctuation toward the negative opinion?

2. Suppose that the bass population in a lake is affected by terrestrial carbon input (falling leaves, etc.) in a way portrayed in the bifurcation diagram below, with r the carbon input and X the bass population density.



(a) Suppose the carbon level is $r = 55$ and the bass density $X = 0.1$. You want to increase the bass population by introducing new bass from bass farm into the lake. Your goal is to get to a density of approximately $X = 0.6$. To reach your goal, up to what density do you need to introduce bass from the bass farm? Explain your reasoning.

(b) Now suppose that you can manipulate the carbon inputs to this system (but you don't introduce bass from a bass farm). If initially, $r = 70$ and $X = 0.05$, how could you manipulate r to raise X to approximately 0.9? Describe how X will change during the manipulations.

(c) Suppose that is hard for you to regulate fishing (and thus the exact density of bass) but that you can control carbon inputs into the lake. If you goal is to preserve a density of at least 0.4, what should you do about the carbon input r?