Recitation 20

Saddle Node Bifurcation: Spruce Budworm Outbreak

Spruce budworms are caterpillars that inhabits forests of the northeastern United States. Usually, they are present in low numbers, but sometimes the population increases dramatically, to the point of defoliating and/or killing large acreages of forest.

This worksheet focuses on analyzing the system and the equilibrium points (in order to build the bifurcation diagram). Next class, we will see and to interpret the bifurcation diagram of this system to answer the questions:

1. Why do these outbreaks happen?
2. How can we manage these outbreaks?

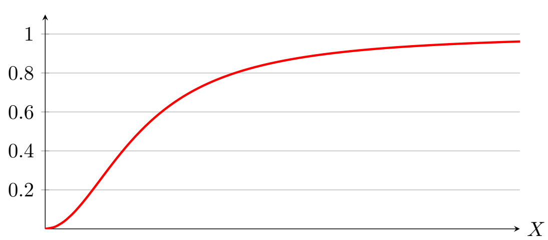
1. We can model the spruce budworm population (X) in a given area by the equation

$$X' = rX\left(1 - \frac{X}{k}\right) - \frac{X^2}{1+X^2},$$

where the first term represents the population growth and the second term predation by birds.

(a) What kind of growth is assumed for the spruce budworm?

(b) The term $\frac{X^2}{1+X^2}$ represents predation of budworms by birds (this is an example of a sigmoid function). Using the graph below, describe in words how the predation depends on the number of budworms.



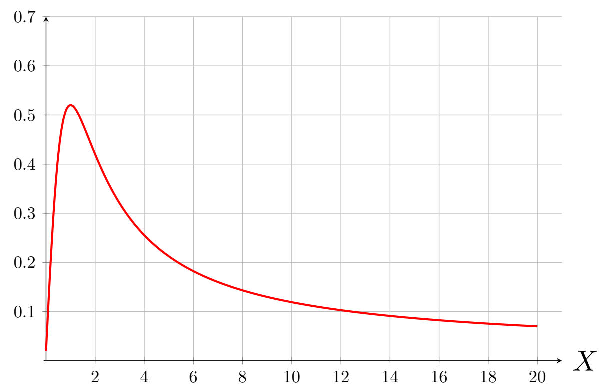
2. In this question, we will solve for the equilibrium points of the system and determine their stability using the "over-under" method.

Factoring the change equation, we have $X' = X\left(r\left(1 - \frac{X}{k}\right) - \frac{X}{1+X^2}\right)$.

Therefore, when we set $X' = 0$, we have either $X = 0$, or $r\left(1 - \frac{X}{k}\right) - \frac{X}{1+X^2} = 0$. Rewriting this latter equation as $r\left(1 - \frac{X}{k}\right) = \frac{X}{1+X^2}$, we see that the equilibrium points are the intersection points of the functions on the left and right sides.

(a) First, we reason about the function on the left side, $Y = r\left(1 - \frac{X}{k}\right)$. This is an equation of a line (make sure you understand why). What are the X- and Y-intercepts of this line?

The graph of $\frac{X}{1+X^2}$ is depicted below.



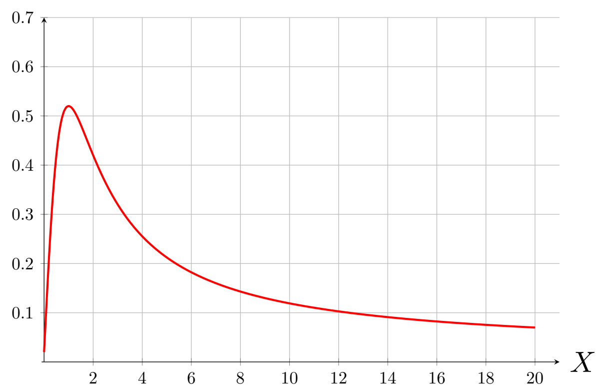
(b) Draw the line (from part (a)) corresponding to $k = 20$ and $r = 0.4$. Locate the equilibria and draw the phase portrait for the system.

How many stable equilibrium points are there? To what population size (small, medium, large) do they correspond?

(c) Answer the same questions when $k = 20$ and $r = 0.15$.

(d) Answer the same questions when $k = 20$ and $r = 0.65$.

(e) Suppose that we fix the carrying capacity at $k = 20$ (as in parts (b)-(d)). For approximately which values of r do we transition between having 1 and 2 stable equilibria? How can we geometrically describe this situation?



3. (a) When $k = 20$, describe the behavior of the system as a function of the parameter r (that is, as we increase r from 0 to 0.7, how does the number of stable equilibria change?).

(b) Answer the same question when $k = 10$. How does this answer differ from part (a)?

(c) Answer the same question when $k = 2$.

Supplemental Questions

A. In question 2(e), we see that values of the bifurcation points are related to the equations of tangent lines. Here, we use our knowledge of tangent lines to exactly solve for these bifurcation points.

(a) From question 2(e), we see that we would like to solve for lines that

- Pass through the point $(20, 0)$.
- Are tangent to $f(X) = \frac{X}{1+X^2}$.

Equivalently, we can find values of X such that the tangent line to $f(X)$ at these X-values pass through $(20, 0)$. Express this criteria as a mathematical equation.

(b) Solve this equation for the desired X-values. (Feel free to use software such as WolframAlpha for the calculations).

(c) Now that we have these X-values, we have located 2 points on each of the tangent lines (the point of tangency and $(20, 0)$). Use this to solve for the values of r at the bifurcation points.