Recitation 17

1. After we solve for the equilibrium points of a system, we have learned about two methods for testing their stability: the *test-points method* and *linear stability analysis*. Write a short description of each of these methods.

2. The spread of a genetic mutation in a population of mice can be modeled by the differential equation

$$P' = 6P^3 - 8P^2 + 2P = 2P(1 - P)(1 - 3P)$$

where P is the fraction of the mice that have the new gene (so $0 \leq P \leq 1$).

(a) Find the equilibrium points of this model.

(b) Draw a phase portrait for the system. Use this to determine the stability of each equilibrium.

(c) Verify that your answer in part (b) is correct using linear stability analysis.

(d) Describe the long term behavior of the system. Be sure your explanation considers every point in the state space.

3. During a consultation, an oncologist estimates the mass of a tumor to be 40 grams. She believes that the growth of the tumor will follow the Gompertz growth model:

$$X' = X\left(0.45 - 0.24\ln\left(\frac{X}{40}\right)\right),$$

where the state variable X corresponds to the mass of the tumor.

(a) Find the equilibrium point(s) for this model.

(b) Determine the stability of the equilibrium point(s).

(c) What should the oncologist expect regarding the long term growth of the tumor.

4. We have seen a special example of the Allee Effect [1] in lecture, here we look into the general case.

We can model the Allee effect by adding a term to the logistic equation. The modified equation becomes

$$X' = rX\left(1 - \frac{X}{k}\right)\left(\frac{X}{a} - 1\right),$$

where we assume that $0 \leq a \leq k$.

(a) Find the equilibrium points of this model.

(b) Determine the stability of the equilibrium points and draw a phase portrait.

(c) What is the long-term behavior of the system? Make sure to describe all possibilities.

(d) Based on your previous analysis, what do the constants a and k represent?

[1] In some species, a minimal number of animals is necessary to ensure the survival of the group. For example, African wild dogs hunt in packs and would not be able to locate and capture prey efficiently if the group is too small. Another possible benefit of being part of a big group is to protect against predation. For example, sardines swim in schools in order to confuse and avoid predators. As a result, their reproductive success declines at low population levels, and a population that is too small may go extinct. This decline in per capita population growth rates at low population sizes is called the Allee effect.