Recitation 16

1. Radioactive iodine (radioiodine) is commonly used in the treatment of thyroid cancer. The thyroid naturally absorbs this iodine, at which point the radiation destroys cancer cells. Moreover, the iodine quickly decays into stable isotopes, making the procedure relatively safe.

(a) We can model the amount of radioiodine (in mg) I present in a cancer patient t days after radiation treatment by the differential equation

$$\frac{dI}{dt} = -0.09I.$$

Separate and integrate to find the general solution to this equation.

(b) Suppose that 1 week after the treatment, the radiologist determines that there are still 5mg of radioiodine present in the patient. Use this information to find the particular solution.

(c) How much radioiodine was initially introduced into the patient?

(d) How long will it take for the amount of radioiodine in the patient to fall below 0.5 mg?

2. A simple population model for rabbits (X) is given by

$$\frac{dX}{dt} = 0.07X,$$

where t is measured in days [1].

(a) Suppose that there are initially 20 rabbits present. Solve this differential equation to express the rabbit population as a function of t (which we think of as the number of days since the initial measurement).

(b) How many rabbits does the model predict will be present after

1 week?

1 month?

1 year?

[1] 0.07 is an actual estimate of the theoretical birth rate of rabbits, which predicts that a pair of rabbits will have an average of 48 bunnies per year!

3. Clearly, the simple model from the previous question is unrealistic. As we learned in Chapter 1, one way that we can improve the model is to add a crowding factor, which enforces an environmental carrying capacity. In our case, we will assume that the environment can support 400 rabbits, which we model by the **logistic differential equation**:

$$\frac{dX}{dt} = 0.07X\left(1 - \frac{X}{400}\right)$$

(a) Show that the general solution to this differential equation is

$$X = \frac{400}{1 + Ae^{-0.07t}},$$

where A is a multiplicative constant of integration. To do this, plug into the differential equation and show that both sides are the same.

(b) Determine the particular solution assuming the same initial condition as in the previous problem ($X(0) = 20$).

(c) Look at a graph of your particular solution using a graphing software such as `desmos.com`. Verify that the function obeys the initial condition, and that it predicts that the population will tend toward the carrying capacity $X = 400$.

Supplemental Questions

A. Determine the general solution to the following separable differential equation:

$$\frac{dX}{dt} - t\sqrt{X} = 0, \qquad X(4) = 9$$

B. Here, we generalize the results from Question 3. Show that the solution to the logistic differential equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{k}\right).$$

with initial condition $X(0) = x_0$ is

$$X = \frac{kx_0}{x_0 + (k - x_0)e^{-rt}}.$$