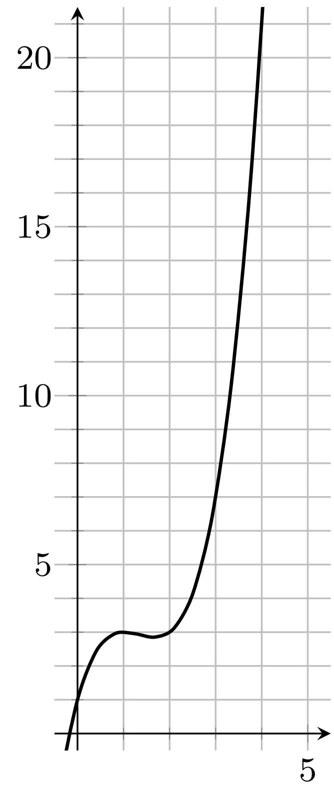
Recitation 13

1. In this question, we explore the quality of a linear approximation to a function.



(a) Find the equation of the tangent line to $f(X) = X^3 - 4X^2 + 5X + 1$, shown on the right, at $X = 2$.

(b) Verify your answer from part (a) by drawing the graph of the line and ensuring that it is, in fact, tangent to $f(X)$ at $X = 2$.

(c) The **approximation error** is a measure of the quality of a linear approximation. Specifically, suppose that $L(X)$ is the linear approximation (tangent line) to $f(X)$ at $X = a$. Then, the approximation error $E(X)$ measures the distance between the actual value of the function and the value of the approximation, $E(X) := |f(X) - L(X)|$.

Calculate the approximation error $E(X)$ for $X = 2$, $X = 2.5$, $X = 3$, and $X = 4$, drawing these errors on the plot above.

Based on these calculations, what can we say about the accuracy of the linear approximation?

2. One application of linear approximations is to approximate the value of "complicated functions" [1]. Here, we explore this phenomenon.

The number of species S living in a habitat with area A (acres) can be modeled as $S = cA^z$, with constants $c, z > 0$. Common values for these constants are $c = 1$ and $z = \frac{1}{2}$, simplifying the function to $S = \sqrt{A}$.

(a) Interpret the meaning of the quantity $\frac{dS}{dA}$. What are its units?

(b) Compute $\frac{dS}{dA}$. And plot this function using a graphing software such as `desmos.com`. Explain the significance on the shape of this function on biodiversity conservation.

A biologist warns a town in Central New York of a decrease in local biodiversity and suggests that the town set aside land for a wildlife preserve. The town locates a 102 acre site, and wants to know how many species this will be able to support. We calculate this value ($\sqrt{102}$) without a calculator.

(c) From Question 1, we know that we can estimate $S(102)$ accurately by finding a linear approximation to $S(A)$ somewhere close to $A = 102$. What would be a good choice for the point of tangency?

[1] In fact, a technique known as **Newton's method** performs calculations via a series of linear approximations. Newton's method forms a basis for most of the computations done on a scientific calculator.

(d) Compute the linear approximation at this point of tangency.

(e) Plug 102 into this linear approximation equation to find your approximation of $\sqrt{102}$. How does it compare to the actual value?

3. The optimal cruising speed V for a bird in flight is given by the **allometric equation**:

$$V = 12M^{1/6},$$

where M is the mass of the bird in kilograms, and V is in meters per second (that is, the bird's speed is a function of its mass).

(a) Based on this equation, would we expect a heavier bird to fly faster or slower than a lighter bird?

(b) Find an expression for the derivative $\frac{dV}{dM}$. How can we interpret the derivative in this context? What are its units?

(c) The average female bald eagle weighs around 5.6 kg and it is about 0.7 kg lighter than a male bald eagle. Using a linear approximation (that is, do not just plug the male weight into the allometric equation), estimate how much faster or slower the optimal cruising speed for a male should be. Give your approximation to two decimal places.

Supplemental Questions

A. Determine the equation of a line tangent to $g(X) = X^2 - 4X + 7$ with slope 4.

B. (*Warning: This question is harder than anything we'd expect you to do. Only attempt if you'd like a challenge.*)

Find the equation of a line that is tangent to both $f(X) = X^2$ and $g(X) = X^2 - 2X + 2$ (at different points).