Recitation 10

1. Compute the following limits:

(a) $\lim_{X\to 2} X^3 - 7X^2 + 4X - 6$

(b) $\lim_{X\to 3} \frac{X^2 + 3X - 6}{X - 2}$

(c) $\lim_{X\to \frac{\pi}{2}} \sin(X)$

(d) $\lim_{X\to -4} \frac{X^2 + X - 12}{X + 4}$

(e) $\lim_{X\to 2} \frac{3X^2 + 15X - 42}{X^2 - 7X + 10}$

2. Compute the following limits. Notice that they are different from the limits above because the values of these limits are functions of X. As we will see next class, this type of limit will be important when we study another fundamental building block of calculus, the **derivative**.

(a) $\lim_{h\to 0} \frac{(X+h)^2 - X^2}{h}$

(b) $\lim_{h\to 0} \frac{\frac{1}{X+h} - \frac{1}{X}}{h}$

3. In this question, we will explore the concept of limits at infinity. Namely, we will consider calculations of the form

$$\lim_{X\to\infty} f(X) \qquad \text{and} \qquad \lim_{X\to-\infty} f(X).$$

(a) Using the definition of limits that you were given in class, what is a reasonable interpretation of the following:

$$\lim_{X\to\infty} f(X) = 4.$$

(b) Compute the following limits at infinity:

$$\lim_{X\to\infty} \frac{1}{X} \qquad \lim_{X\to\infty} 2 - X^2$$

$$\lim_{X\to-\infty} e^X \qquad \lim_{X\to\infty} \sin(X)$$

Next, we explore limits at infinity of **rational functions**, functions which can be expressed as a quotient of two polynomials,

$$f(X) = \frac{p(X)}{q(X)}.$$

We'll come up with rules for these limits by looking at the graphs of these functions. You may wish to use a graphing calculator, or a website such as `desmos.com`.

(c) First, we consider the case where the degree of the numerator $p(X)$ is **larger** than the degree of the denominator $q(X)$. Examples of such function are

$$f(X) = \frac{X^2 + 3X}{X + 2}, \qquad \frac{X^3}{X^2 - 4X}, \qquad \frac{X^7}{X^3 + X^2 + X + 1}$$

By looking at the graphs of these functions, what is $\lim\limits_{X\to\infty}$ in this case?

(d) Next, we consider the case where the degree of the numerator $p(X)$ is smaller than the degree of the denominator $q(X)$. Examples of such function are

$$f(X) = \frac{X}{X^2 - 1}, \qquad \frac{1}{X^2 - 2}, \qquad \frac{X^3}{X^4 - X^2 + 7}$$

By looking at the graphs of these functions, what is $\lim\limits_{X\to\infty}$ in this case?

(e) Finally, we consider the case where the degree of the numerator $p(X)$ is the same as the degree of the denominator $q(X)$. This case is a little trickier because the answer depends on the leading coefficients [1] of $p(X)$ and $q(X)$. Looking at the graphs of the following functions,

$$f(X) = \frac{X^2+3}{X^2-X}, \qquad \frac{2X}{3X+4}, \qquad \frac{6X^3-5X}{2X^3+17}$$

what is the rule for calculating $\lim_{X\to\infty}$ in this case?

(f) Use your answers from parts (c)-(e) to compute the following limits at infinity:

$$\lim_{X\to\infty} \frac{-4X^3+3X-16}{7X^3-X^2} \qquad \lim_{X\to\infty} \frac{3-2X+X^4}{3X^2-7}$$

$$\lim_{X\to\infty} \frac{2X^2-6}{18-X^7} \qquad \lim_{X\to\infty} \frac{12X^3-16X+4}{36X-4X^2+18X^3}$$

4. Two special families of rational functions are the increasing and decreasing **sigmoid** (S-shaped) functions. These functions will be important later in the course when we use them to improve the Lotka-Volterra predation model.

(a) An increasing sigmoid function has the form

$$f(X) = \frac{c(a+X^n)}{d+X^n}$$

where n is a positive integer and $a, c, d > 0$ are constants.

Explain, in terms of limits, the significance of the constant c. (Hint: Use Question 2e.)

(b) A decreasing sigmoid function has the form

$$g(X) = \frac{a}{d+X^n}$$

where n is a positive integer and $a, d > 0$ are constants. What is $\lim_{X\to\infty} g(X)$?

[1] As a reminder, the leading coefficient of a polynomial is the coefficient on the highest degree term. For example, the leading coefficient of $f(X) = 2X^2 - 3X^3 + 4X - 6$ is -3.

Supplemental Questions

A. Consider the function $f(X) = \frac{-2X^2+14X-24}{X^2-3X-4}$. Note that this function is not defined (and therefore not continuous) at $X = 4$. This is an example of a **removable** discontinuity. To remove the discontinuity, we will add a definition at $X = 4$ to the definition of f, so that

$$f(X) = \begin{cases} \frac{-2X^2+14X-24}{X^2-3X-4} & X \neq 4 \\ a & X = 4 \end{cases}$$

for some constant a.

For what value of a is $f(X)$ continuous at 4?

B. Determine the rule for computing $\lim\limits_{X\to-\infty}$ in the case of Question 2c (when $f(X)$ is a rational function such that the numerator has higher degree than the denominator). As a hint, this will depend on the degrees.