Propositional Logic

Review

A **proposition** is a statement that can be assigned a truth-value; that is, a statement that is either true or false.

We can use **logical operators** in order to build more complicated (compound) propositions from simpler (atomic) ones. Consider propositions P and Q. We have,

$$\begin{aligned} \text{Negation } (\neg): &\quad \neg P \text{ has the opposite truth value of } P. \\ \text{Conjunction } (\wedge): &\quad P \wedge Q \text{ is true only when both } P \text{ and } Q \text{ are true.} \\ \text{Disjunction } (\vee): &\quad P \vee Q \text{ is true as long as at least 1 of } P, Q \text{ is true.} \\ \text{Implication } (\implies): &\quad P \implies Q \text{ is true unless } P \text{ is true and } Q \text{ is false.} \\ \text{Bidirectional Implication } (\iff): &\quad P \iff Q \text{ is true when } P \text{ and } Q \text{ have the same truth value.} \end{aligned}$$

We can use **truth tables** to visualize the truth value of a compound proposition based on the truth value of its constituent propositions. Truth tables for the above operators are shown below.

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

In the first columns of the truth table (one per atomic proposition), we enumerate the possible combinations of truth values of the atomic propositions. In each subsequent column, we build up more complicated propositions by applying one of the operators, until we are left with the desired proposition in the rightmost column.

A compound proposition is also called a **formula**.

A formula is **satisfiable** if there is some assignment of truth values to its atomic propositions which makes the formula true.

A formula is a **tautology** if every assignment of truth values to its atomic propositions makes the formula true.

A formula is a **contradiction** if no assignment of truth values to its atomic propositions makes the formula true.

Two formulas are **(logically) equivalent** if for every assignment of truth values to their constituent atomic propositions, the formulas have the same truth value.

DeMorgan's Laws give two commonly-used logical equivalences:

$$\neg(P \wedge Q) = \neg P \vee \neg Q$$
$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

1. The exclusive-or (xor) operation, represented with the symbol $\oplus$, evaluates to true when exactly one of its arguments is true.

(a) Write out the truth table for exclusive-or.

(b) Write down a formula for exclusive-or that uses only $\wedge, \vee$, and $\neg$.

(c) The negation of exclusive-or is equivalent to which other logical connective that we have talked about?

$\neg(P \oplus Q) \equiv P \wedge Q$ $\neg(P \oplus Q) \equiv P \vee Q$

$\neg(P \oplus Q) \equiv P \implies Q$ $\neg(P \oplus Q) \equiv P \iff Q$

2. The boolean operations are realized in computer hardware by logical gates, which take in two input voltages (F = low voltage, T = high voltage) and output the voltage corresponding to their logical operation (so an AND gate puts out a high voltage exactly when both of its input voltages are high).

In many cases, these logic gates are sold on chips which contain many copies of the same gate. Therefore, to save money, it can often be desirable to rewrite a boolean expression to use fewer *types* of logical operations, even if this increases the *number* of operations.

(a) Explain how you can transform a formula including $\wedge, \vee$, and $\neg$ into a formula involving $\wedge$ and $\neg$ only.

(b) Use your answer from part (a) to rewrite the following expressions using only $\wedge$ and $\neg$. For the latter expressions, it may be helpful to first express them in terms of $\wedge, \vee$ and $\neg$.

$$\neg P \vee \neg Q :$$

$$P \vee Q \vee R :$$

$$P \implies Q :$$

$$P \iff Q :$$

(c) **Challenge:** The logic gate NAND, sometimes represented by the symbol $\overline{\wedge}$, is defined to be the negation of $\wedge$; that is, $P \mathbin{\overline{\wedge}} Q = \neg(P \wedge Q)$. Express the following three logical expressions using only $\overline{\wedge}$.

$$\neg P :$$

$$P \wedge Q :$$

$$P \vee Q :$$

Taken together, parts (b) and (c) show that every logical operation that we have studied can be expressed using only $\overline{\wedge}$ gates.

3. An operation $\diamond$ is *associative* if $A \diamond (B \diamond C) = (A \diamond B) \diamond C$ for any arguments A, B, C; that is, we obtain the same result regardless of the order that the operation is applied to a sequence of arguments.

Determine whether $\implies$ is associative.

4. Consider the following 3 logical propositions.

$$T := \text{The temperature outside is over 75 degrees.}$$
$$P := \text{I spend the afternoon at the pool.}$$
$$I := \text{I stop for ice cream.}$$

Provide a symbolic expression that captures the logical meaning of the following sentences.

(a) The temperature did not go above 75 degrees today, but I still spent the afternoon at the pool.

(b) On any hot day (a day where the temperature exceeds 75 degrees), I either go to the pool in the afternoon or I get ice cream.

(c) I only stop for ice cream when it is over 75 degrees out.

(d) I never go to the pool in the afternoon when it is under 75 degrees out.

(e) When it's over 75 degrees out, I must stop for ice cream on days that I go to the pool.