Proof Techniques

Review

A **proof** is a chain of logical deductions building on a set of assumptions or axioms to verify a proposition.

When we write proofs, we try to make them easy for the reader to follow by:

1. Writing out ideas in complete sentences.
2. Using words and explanations instead of too many symbols and equations.
3. Including all steps of our reasoning to avoid any logical leaps.

There are many standard **proof techniques**, common structures that the argument in a proof can take. Certain techniques will work better for certain situations, so it is important to be comfortable with all of the techniques.

Direct Proof: Argues a proposition by a sequence of logical steps. Each assertion in the proof is a consequence of the previous assertions by means of a definition, one of our assumptions, or an algebraic law (some mathematical property from high school or earlier).

Proof by Case Analysis: A direct proof, where we reason separately about various situations. These proof are useful when there are a small number of cases to handle, or when breaking down the problem gives us extra properties to use in our proof. It is important to ensure that the cases that you consider cover all possible scenarios.

Contrapositive Proof: When we are trying to prove an implication ($P \implies Q$), it is often easier to prove its contrapositive ($\neg Q \implies \neg P$), which we know is logically equivalent.

Proof by Contradiction: This is an indirect proof technique. We assume that the proposition that we are trying to argue is false. Then, we use a sequence of assertions to show that this assumption leads to a contradiction (either to some other assumption we have made or to some other mathematical law/property that we know to be true).

Later in the course, we will learn about another proof technique, **induction**, which is useful for proving statements that are quantified over the natural numbers.

In order to prove that a claim is not true, it suffices to give a **counterexample**, one specific instantiation of any variables in the claim for which the desired assertion does not hold.

In our early introduction to proof techniques, we make use of the following (hopefully familiar) properties of numbers.

- A positive integer is **prime** when its only positive integer factors are 1 and itself. Integers greater than 1 that are not prime are said to be **composite**.
- An integer n is **even** when it can be written as $n = 2j$ for some integer j. Otherwise, n is **odd** and can be written as $n = 2k + 1$ for some integer k.
- A number n is **rational** if it can be expressed as $n = \frac{p}{q}$ for integers p and q. Otherwise, it is **irrational**.

1. Prove that if n is an even integer with $4 \leq n \leq 20$, then n can be written as the sum of 2 prime numbers. (Recall that a positive integer is *prime* if its only positive integer factors are 1 and itself.)

(a) Which of the following proof techniques makes most sense to use here? Give a sentence justifying why.

Proof by Contradiction **Case Analysis** **Contrapositive Proof**

(b) Write a proof of this claim using your chosen method.

Proving this for all even $n \geq 4$ would verify *Goldbach's Conjecture*, a famous unsolved problem in mathematics.

2. Find a counterexample to disprove each of the following claims:

(a) For all non-negative integers n, $2^n + 3^n + n(n-1)(n-2)$ is prime.

(b) Every odd integer $n \geq 5$ can be written as the sum of 2 prime numbers.

(c) Given integers x, y, if $x < y$ and $y > 0$, then $x^2 < y^2$.

3. Given any integer x, argue that $3x^2 + 7x - 20$ is even.

(a) Which of the following proof techniques makes most sense to use here? Give a sentence justifying why.

Proof by Contradiction **Case Analysis** **Contrapositive Proof**

(b) Write a proof of this claim using your chosen method.

4. Argue that given any three integers, two of them sum to an even number. Be sure to state which proof technique you are using.

5. Consider the following claim:

Suppose that $x^3 + 3x + 4 < 0$. Then x must be negative.

(a) Express this claim as an implication.

(b) What is the contrapositive of this implication.

(c) Which of these implications ((a) or (b)) is easier to argue? Explain why.

(d) Use your answer to part (c) to select a proof technique. Use this to prove the claim.

6. Consider the following false proof that $2 = 1$:

Proof. Let a and b be equal, non-zero integers. Then,

$$
\begin{aligned}
& & a &= b & (1)\\
&\text{Multiply by } a: & a^2 &= ab & (2)\\
&\text{Subtract } b^2: & a^2 - b^2 &= ab - b^2 & (3)\\
&\text{Factor}: & (a+b)(a-b) &= b(a-b) & (4)\\
&\text{Divide by } (a-b): & a+b &= b & (5)\\
&\text{Substitute } a = b: & b+b &= b & (6)\\
&\text{Combine like terms}: & 2b &= b & (7)\\
&\text{Divide by } b: & 2 &= 1 & (8)
\end{aligned}
$$

Therefore, $2 = 1$. $\square$

Locate the mistake in this proof.

7. Optional: Suppose that x is a rational number and y is irrational. Argue that $x + y$ is irrational.

8. Optional: Prove that $\log_2 6$ is irrational. Your proof may use the fact that each positive integer can be uniquely expressed as a product of prime numbers.

As a reminder, $\log_b a$ evaluates to the exponent which one must raise b to in order to obtain a. That is, $\log_b a = c$ means that $b^c = a$.