Induction

Review

Induction is a proof technique that is used to argue that some property holds for all (or in some cases, a pertinent subset) of the natural numbers. To use induction we,

1. Define an **inductive statement** $P(n)$, a predicate with argument domain n ranging over the natural numbers.
2. As a **base case**, argue that $P(n)$ is true for the smallest case(s), usually $n = 0$ or $n = 1$.
3. As an **inductive hypothesis**, assume that $P(n)$ is true for some arbitrary (fixed) n, greater than or equal to the base case.
4. For the **inductive step**, use the inductive hypothesis to argue that $P(n + 1)$ is also true.

Once we have shown a base case and the inductive step, we may conclude that $P(n)$ is in fact true for all natural numbers greater than or equal to the base case.

It is essential that this process, in its entirety, is followed when writing an inductive proof.

Generally, the trickiest part of an inductive proof is figuring out how to apply the inductive hypothesis during the inductive step.

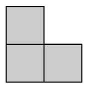
A related technique is **Strong Induction** (sometimes called **Complete Induction**). Here, we replace the inductive hypothesis with a stronger statement. Namely, for inductive step, we seek to prove that $P(n+1)$ is true under the assumption that $P(k)$ is true for all $b \leq k \leq n$, where b is the smallest value for which $P(b)$ is proven in the base case(s).

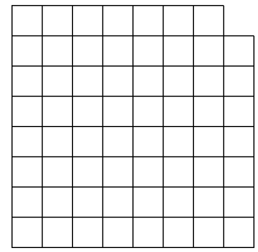
1. Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$.

2. Use induction to argue that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \geq 1$.

3.

Consider a $2^n \times 2^n$ ($n \geq 1$) square grid with the upper right square removed, such as the 8×8 grid shown on the left. We'll argue by induction that such a grid can be completely covered by these L-shaped tiles:





(a) What is the inductive statement $P(n)$ that we are trying to prove?

(b) The base case for our proof is immediate. Why is this?

(c) What do we take for our inductive hypothesis?

(d) For the inductive step, what must we show to complete our proof?

(e) Perform the inductive step to complete the proof. (**Hint:** Your argument should invoke the inductive hypothesis. How can we apply our fact about $2^n \times 2^n$ grids to our $2^{n+1} \times 2^{n+1}$ grid?)

4. Consider the same tiling problem as in the previous question.

(a) Write an inductive definition for t_n, the number of L-shaped tiles required to cover the $2^n \times 2^n$ board.

(b) Determine a closed formula for the number of tiles needed to cover the $2^n \times 2^n$ grid. (**Hint:** How many total squares are in this grid?)

(c) Use induction to verify that your closed formula from part (b) is the solution to your inductive definition in part (a).

5. In the 1980s, McDonald's sold Chicken McNuggets in packs of 6, 9, and 20. A number $n \in \mathbb{N}$ is called a *McNugget number* if it is possible to order exactly that many McNuggets from the menu. For example, 15 is a McNugget number because you can get exactly 15 McNuggets by ordering a 6-pack and a 9-pack. Similarly, a number is called a *non-McNugget number* if that many McNuggets cannot be ordered.

(a) Argue that 43 is a non-McNugget number.

We'll use complete induction to argue that 43 is the largest non-McNugget number.

(b) What should we take for our inductive statement $P(n)$?

(c) How many base cases should we consider? What are they (with proof)?

(d) Complete the proof by stating an inductive hypothesis, and using this to carry out the inductive step.

(e) **Challenge:** Later, McDonald's introduced a children's 4 pack of McNuggets. What is the new largest non-McNugget number.

6. Verify that $\left(\sum_{x=1}^{n} x\right)^2 = \sum_{x=1}^{n} x^3$ by induction on n.

You may use the fact that $\sum_{x=1}^{n} x = \dfrac{n(n+1)}{2}$ without proof.