Functions

Review

Given two sets, a **domain** set D and a **codomain** set C, a **function** $f : D \to C$ represents a mapping from D to C. That is, f assigns each domain element $d \in D$ to a codomain element $c \in C$.

We can therefore view f in two ways:

1. $f \subseteq D \times C$, a subset of all pairs of a domain and codomain element (so $(d, c) \in f$ above).
2. f is a rule telling us where in the codomain to send each domain element (so $f(d) = c$ above).

The first viewpoint is more formal, while the second viewpoint is probably more familiar. We can visualize a function using an arrow diagram, such as those on the following page, with the domain pictured on the left and the codomain on the right.

A function is **well-defined** when each domain element maps to a unique element of the codomain. Sometimes, one studies **partial** functions, which map only a subset of the domain (we don't consider partial functions in this course). In this case, the word **total** is used to distinguish (partial) functions that map their entire domain. For our purposes, we consider a function well-defined only if it is total.

The **image** of a domain element $d \in D$ under function f is the codomain element $c \in C$ such that $f(d) = c$ (that is, where f maps d to).

The **pre-image** of a codomain element $c \in C$ under f is the **subset** $S \subset D$ such that $f(s) = c$ for each $s \in S$ (that is, where f mapped to c from). A function is **injective** (1 to 1) when the pre-image of each codomain element has size at most 1 (no two domain elements map to the same element of C).

The **range** of a function is the subset of the codomain which is mapped to (the collection of all images of $d \in D$). A function is **surjective** if its range equals its codomain (it maps to each element of the codomain).

A function that is both injective and surjective is called **bijective**.

Given functions $f : A \to B$ and $g : B \to C$ (that is, the codomain of f is the same as the domain of g), we can define the **composite** function $g \circ f : A \to C$ (read "g of f") with $(g \circ f)(a) = g(f(a))$ for each $a \in A$.

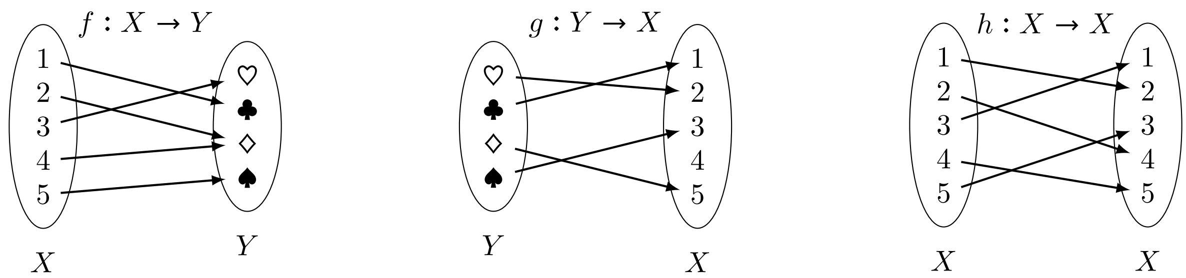
The **identity** function on a set A, $\mathbf{id}_A$, maps each element to itself. A function $g : B \to A$ is a **left inverse** of $f : A \to B$ if $g \circ f = \mathbf{id}_A$. It is a **right inverse** of f if $f \circ g = \mathbf{id}_B$. If g is both a left and right inverse of f, then it is the (unique) **inverse** of f (we think of g as the reverse mapping of f). A function is **invertible** if it has an inverse. A function is invertible if and only if it is bijective.

We say that $|S| \le |T|$ if there is an injective function $f : S \to T$.

We say that $|S| = |T|$ if $|S| \le |T|$ and $|T| \le |S|$, so there is an injective function $f : S \to T$ and an injective function $g : T \to S$. The Schröder-Bernstein theorem allows us to conclude that, in this case, there must be a bijection $h : S \to T$.

A set S is **countable** if $|S| \le |\mathbb{N}|$ and **countably infinite** if $|S| = |\mathbb{N}|$. If $|S| \not\le |\mathbb{N}|$, then S is **uncountable**.

1. Consider the following functions, represented as arrow diagrams:



(a) Draw the arrow diagrams for the composite functions:

$f \circ g$ $g \circ f$

(b) Circle the functions that are...

Injective:	f	g	h	$f \circ g$	$g \circ f$
Surjective:	f	g	h	$f \circ g$	$g \circ f$
Bijective:	f	g	h	$f \circ g$	$g \circ f$

(c) Determine whether each of the following compositions of functions makes sense. For those that do, what are their domain and codomain? For the functions with domain X, determine where they send 3. For the functions with domain Y, determine where they send ♣.

$f \circ h$:

$g \circ h$:

$f \circ h \circ g$:

$h \circ h \circ h$:

2. How many functions are there from $X = \{a, b, c\}$ to $Y = \{1, 2\}$? Write them all down.

3. Suppose that $f : X \to Y$ represents a mapping from domain X and codomain Y. Match each of the properties on the left to its equivalent representation with logical quantifiers on the right. One of the statements on the right will not be used.

f is a (well-defined) function.

f is injective.

f is surjective.

$\forall\ y \in Y\ \exists\ x \in X.\ f(x) = y$

$\forall\ x \in X\ \exists\ y \in Y.\ f(x) = y$

$\forall\ x_1 \in X\ \forall\ x_2 \in X.\ \Big(f(x_1) = f(x_2) \implies x_1 = x_2\Big)$

$\forall\ x \in X.\ \Big|\{y : f(x) = y\}\Big| = 1$

4. Suppose that $f : X \to Y$ and $g : Y \to Z$ are both bijections, and define $h = g \circ f$.

(a) What are the domain and codomain of h?

We'll argue that h is also a bijection. We split this into two separate arguments: first that h is injective, and second that h is surjective.

(b) Argue that h is injective.

(c) Argue that h is surjective.

(d) The converse of the claim that we just proved is:

> *Suppose that $h : X \to Z$ is bijective and $h = g \circ f$ for some functions $f : X \to Y$ and $g : Y \to Z$. Then, f and g are also bijections.*

Give an example that shows that this converse is not true.

5. In this question, you'll prove one of the theorems that was stated in lecture:

A function is left invertible if and only if it is injective.

To introduce some terminology we'll use in our proof, we'll consider a function $f : X \to Y$; that is, f has domain X and codomain Y.

(a) To start, we'll prove the forward implication. For this, we assume that f is left invertible, so it has a left inverse g.

Describe g. What are its domain and codomain? What additional property does g have as the left inverse of f?

(b) We must argue that f is injective. Appealing to the definition of injectivity, we'll suppose that $f(x_1) = f(x_2)$. From here, we must argue that $x_1 = x_2$. Complete this argument. (**Hint:** You'll need to make use of g.)

(c) Next, we argue the reverse implication. We'll suppose that f is injective. Use this to construct a left inverse g for f, and verify that this g satisfies the property from part (a).

6. In this question, we'll argue that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

(a) First, argue that $|\mathbb{N}| \leq |\mathbb{Z}|$ by exhibiting an injective function $f : \mathbb{N} \to \mathbb{Z}$. (**Hint:** There's a simple function that works.)

(b) Next, argue that $|\mathbb{Z}| \leq |\mathbb{N}|$ by exhibiting an injective function $g : \mathbb{Z} \to \mathbb{N}$.

To get you started, note that taking $g(0) = 0, g(1) = 1, g(2) = 2, \cdots$ will not work because we'll "fill" up the natural numbers with the non-negative integers, leaving no space for us to map the negative integers. You'll need to find a way to expand in both directions in the same time. Write out the the mapping of smaller integers, and then generalize any patterns that show up to find a formula for g.