Combinatorics 2

Review

There are many identities which describe relationships between the binomial coefficients $\binom{n}{k}$. One such identity gives the symmetry,

$$\binom{n}{k} = \binom{n}{n-k}.$$

While it is often straightforward to prove such identities algebraically (using the factorial definition of the binomial coefficients), it is usually more informative to provide a **combinatorial proof**. In such a proof, we show that both sides of the identity are equal by arguing that they are two different ways to count the same collection. A combinatorial proof of the above identity is that both sides count the number of subsets of size k from n elements: we can either select the k elements to include in the set (left side) or select that $n-k$ elements to omit from the set (right side).

Another useful identity is **Pascal's Identity**:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The **Binomial Theorem** gives us a way to more easily expand polynomial expressions.

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

The **Inclusion-Exclusion** rule gives us a way to count the number of objects which have multiple, potentially overlapping, characteristics. The simplest form of the rule is for 2 sets, which gives $|A \cup B| = |A| + |B| - |A \cap B|$, which can be verified by looking at a Venn diagram. For a more general collection of sets $A_1, A_2, \ldots, A_k$, we have,

$$\Big|\bigcup_{i=1}^{k} A_i\Big| = \sum_{\substack{S \subseteq \{1,\ldots,k\} \\ S \neq \emptyset}} (-1)^{|S|-1} \Big|\bigcap_{i \in S} A_i\Big|.$$

Another common special case of the Inclusion-Exclusion rule is when $k=3$. Here, we have,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

In a **balls and bins** problem, we wish to count the number of ways that we can place n balls into m distinguishable bins.

If the balls are distinguishable, this amounts to selecting one of the m bins for each ball. The multiplication rule gives m^n possible arrangements.

If the balls are indistinguishable, each arrangement depends only on the number of balls placed in each bin. We can exhibit a bijection between the number of arrangements and the number of binary strings of length $n + m - 1$ with $m - 1$ 1's (here, the 1s delimit the bins, and the 0s represent the balls). Therefore, the number of arrangements is $\binom{n+m-1}{m-1}$.

The **Pigeonhole Principle** states that any function with a larger domain than codomain must map 2 domain elements to the same codomain element. Colloquially, placing $n + 1$ pigeons into n holes results in at least 2 pigeons in the same hole.

More generally, if $f : D \to C$ is a function with $|D| > k|C|$, then there is some $c \in C$ with $|f^{-1}(c)| > k$.

To apply the Pigeonhole Principle, it is important to specify what the "pigeons" are and what the "holes" are in your counting scenario.

1. A fruit stand carries apples, bananas, and pears. At the stand, you can choose to purchase a fruit basket, which includes eight pieces of fruit of your choice. In this question, we'll determine the number of possible baskets under various conditions. Note that baskets are equivalent if they contain the same number of each type of fruit; the order of the fruits in the basket is irrelevant.

(a) First, we consider the case with no additional conditions (that is, the basket can have any number of each of the fruits, as long as there are eight pieces in total). Describe a bijection between the possible fruit baskets and the binary strings of length ten containing two 1s.

(b) Using your answer from part (a), how many such baskets of fruit are there?

(c) Next, we impose the condition that each basket must contain at least one of each type of fruit. Describe a bijection between these baskets of fruit and the set of fruit baskets containing (any) five pieces of fruit.

(d) Use your answers from the previous parts to determine the number of possible fruit baskets under the condition from part (c).

(e) Finally, suppose that the fruit stand is low on pears and says that each basket can contain at most one pear (and the condition from part (c) no longer applies). How many different baskets are possible now? (**Hint:** Consider fixing the number of pears first, and counting these cases separately. Then, use the addition rule.)

2. Consider the combinatorial identity,

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}.$$

(a) Argue that this identity is correct algebraically (using the definition of binomial coefficients).

(b) Give a combinatorial argument for this identity.

(**Hint:** Consider different ways of choosing a team of k students from a class of size n and designating one of these students the team captain.)

3. Use the binomial theorem to determine the coefficients on the following terms in the respective polynomials.

(a) x^3y in $(x+y)^4$

(b) x^5y^2 in $(x+y)^8$

(c) x^4y^7 in $(x+y)^{11}$

(d) x^2y^3 in $(2x+y)^5$ (Be careful!)

(e) x^8y^2 in $(x^2+y)^6$ (Be careful!)

4. How many integers between 1 and 60 are a multiple of 3 or 5? (**Hint:** Use the Inclusion-Exclusion Principle)

5. Let S be the set of strings of length 8 made up of the characters $\{a, b, c\}$.

(a) How many strings in S contain exactly 3 a's?

(b) How many strings in S contain exactly 3 of one letter? (**Hint:** Use Inclusion-Exclusion and part (a).)

6. Use the Pigeonhole Principle to argue that any set of $n+1$ distinct integers from $\{1, 2, \ldots, 2n\}$, contains two consecutive integers. Be sure to specify what you are using as the pigeons and what you are using as the holes.

7. In this question, we'll prove the following result using the Pigeonhole Principle:

> *In any undirected graph (on at least 2 vertices) with no self-loops, there are two vertices that have the same degree.*

To argue the claim, we'll consider a graph with n vertices (for some arbitrary $n \geq 2$).

(a) First, suppose that the graph contains no isolated vertices; that is, every vertex is incident to at least one edge. Argue the claim in this case.

(b) Next, suppose that the graph contains an isolated vertices. Note that no vertex in the graph can have degree $n-1$, and use this to argue the claim in this case.

(c) Does the claim still hold if we allow self-loops?