# Combinatorics 1

## Review

In **combinatorics**, we aim to determine how many objects obey a certain property without explicitly enumerating them.

The **product rule** states that if there are $|A|$ possibilities for $A$, and for each of these there are $|B|$ possibilities for $B$, then there are $|A| \cdot |B|$ possibilities for $(A, B)$. This rule easily generalizes to a greater number of steps.

We typically use the product rule to model a scenario where we can fix certain attributes of an object sequentially;

The **sum rule** states that if we can break down the objects in $A$ into two *disjoint* subsets $A_1$ and $A_2$. Then the number of objects in $A$, $|A| = |A_1| + |A_2|$. Again, this generalizes to a partition into more sets.

The **bijection rule** states that if there is a bijection between finite sets $A$ and $B$, then $|A| = |B|$.

Suppose that we can establish a $k$-to-1 correspondence between sets $A$ and $B$ (that is, exactly $k$ elements of $A$ map to each element of $B$). Then, the **division rule** states that $|A| = k \cdot |B|$.

In many instances, multiple rules will be needed to determine the size of a collection of objects.

A **permutation** of $r$ out of $n$ objects is an ordered arrangement of $r$ objects, taken from the set of $n$ distinct objects. We use $P(n, r)$ to denote the number of such permutations.

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1) \cdots (n-r+1).$$

A **combination** of $r$ out of $n$ objects is an unordered collection (subset) of $r$ objects, taken from the set of $n$ distinct objects. We use $C(n, r)$ or $\binom{n}{r}$ to denote the number of such combinations.

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

**1.** A sub shop sells child-sized subs, 6-inch, and 12-inch subs. They have a selection of 7 different proteins, 4 different cheeses, 12 toppings, and 5 sauces.

(a) A child-sized sub comes with a choice of protein and cheese, 1 topping, and 1 sauce. How many different child-sized subs can be ordered?

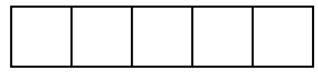
(b) A 6-inch sub comes with a choice of protein and cheese, (exactly) 3 toppings, and 1 sauce. How many different 6-inch subs can be ordered?

(c) Suppose that instead you can order **up to** 3 toppings on a 6-inch sub. Now, how many subs are possible?

(d) A 12-inch sub comes with a choice of protein and cheese, and as many toppings and sauces as you'd like. How many different 12-inch subs can be ordered?

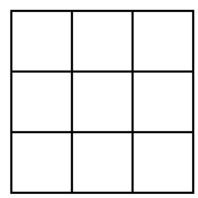
(e) Suppose that the sub shop allows you to order different things on each half of the 12-inch sub. Now how many different 12-inch subs are possible?

**2.** (a) **Warm-up:** Consider a row of five boxes, such as the one shown below.



How many ways are there to shade in a contiguous group of (1 or more) boxes?

Consider a $3 \times 3$ square grid, such as the one shown below.



(b) How many different rectangles (including squares) can be formed by shading cells in this grid?

(c) Let's generalize our result. How many different rectangles (including squares) can be formed by shading cells in an $n \times n$ grid? (**Hint:** Think about choosing the vertical and horizontal bounds of the rectangle separately.)

This question gives an example of a situation where combinatorial reasoning provides a much cleaner argument than an inductive proof can.

**3.** Consider the set $S$ of strings (sequences) of length 6 made from the characters $\{a, b, c\}$. Determine the cardinality of each of the following subsets of $S$, referencing which combinatorial rules you are using.

(a) $S_1 = \{s \in S : s \text{ contains only } a\text{'s and } c\text{'s}\}$

(b) $S_2 = \{s \in S : s \text{ contains exactly 1 } b\}$

(c) $S_3 = \{s \in S : s \text{ contains exactly 2 } b\text{'s}\}$

(d) $S_4 = \{s \in S : s \text{ contains at most 2 } c\text{'s}\}$

(e) $S_5 = \{s \in S : s \text{ is a palindrome}\}$.

Recall that a palindrome is a string that reads the same forwards and backwards.

(f) $S_6 = \{s \in S : \text{ no 2 consecutive letters in } s \text{ are the same}\}$

(g) $S_7 = \{s \in S : \text{ all } a\text{'s in } s \text{ are adjacent}\}$

(**Hint:** You'll need to break this into cases. How should we do this?)

**4.** Consider sets $X = \{1, 2, 3\}$ and $Y = \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}$.

(a) How many functions $X \to Y$ are there?

(b) How many functions $Y \to X$ are there?

(c) How many of the functions $X \to Y$ are injective?

(d) **Challenge:** How many of these functions $Y \to X$ are surjective? (Be careful!)

**5.** An anagram of a word is a rearrangement of its letters, so TOPS, PTOS, and OPTS are all anagrams of STOP.

(a) How many anagrams are there of NEWYORK?

(b) How many anagrams are there of GEORGIA?

(c) How many anagrams are there of INDIANA?

(d) How many anagrams are there of MISSISSIPPI?