Sets

Review

A **set** is a collection of objects, called **elements**, which are unordered and unique (that is, the set contains no duplicate elements). We typically use uppercase letters to denote sets and lowercase letters to denote their elements.

We can notate the elements of a set in two ways. The simpler way, used for smaller sets, is to write a comma-separated list of the elements, enclosed in curly brackets, such as

$$\{1, 2, 3, 4, 5\}.$$

The second way is so-called **set-builder** notation, where we express a set as those elements of a larger set which obey a certain property (make a predicate true). For example, we can rewrite the above set in set-builder notation as

$$\{n \in \mathbb{N} : 1 \leq n \leq 5\},$$

the non-negative integers between 1 and 5 (inclusive).

We can use set operations to build new sets from existing sets. For example, given sets A and B, we have,

Intersection ($\cap$) :	$A \cap B$ contains the elements that belong to both A and B.
Union ($\cup$) :	$A \cap B$ contain any elements that belong to either A or B (or both).
Difference ($\setminus$) :	$A \setminus B$ contains those elements of A that do not belong to B.

There are two operators that we can use to form propositions from sets:

Inclusion ($\in$) :	$a \in A$ is true when a is an element of the set A.
Subset ($\subseteq$) :	$A \subseteq B$ is true when every element of A is also an element of B.

In order to prove that $A \subseteq B$, we often choose an arbitrary element of A and show that it is in B (that is, we show that membership in B is implied by the properties defining set A).

Two sets are equal ($A = B$) when they have the same elements. This is equivalent to both sets being subsets of each other ($A \subseteq B$ and $B \subseteq A$). Therefore, we often split proofs of equality into two subset arguments.

The **cardinality** of a set (denoted $|S|$) is a representation of its number of elements. This is straightforward when the set is finite (the cardinality is in $\mathbb{N}$). We'll consider infinite cardinalities more later.

Finally, we have two more operations which allow us to again build more complicated sets from simpler sets:

Power Set ($\mathcal{P}$) :	The power set of A, $\mathcal{P}(A)$ is the set of all subsets of A.
Cartesian Product ($\times$) :	$A \times B$ consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

1. Given sets:

$$\begin{aligned} A &:= \{2, 5, 7, 3, 1\}, \\ B &:= \{2, 4, 6, 8\}, \\ C &:= \{1, 7, 6\} \end{aligned}$$

write out the elements of each of the following sets.

(a) $A \cup B$:

(b) $A \cap C$:

(c) $B \setminus C$:

(d) $A \cap (B \cup C)$:

(e) $(A \cap B) \cup C$:

(f) $\mathcal{P}(C)$:

(g) $\mathcal{P}(A \cap B)$:

(h) $B \times C$:

(i) $A \times \emptyset$:

2. Given any sets A and B, prove that $(A \setminus B) \cap (B \setminus A) = \emptyset$. (**Hint:** Use a proof by contradiction.)

3.

(a) Given any sets A, B, X, prove the following equivalence,

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

The claim we have proven is one of the "deMorgan's Laws for Sets". To understand this, consider the predicates

$$P(x) := x \in A \qquad\qquad Q(x) := x \in B,$$

with domain $x \in X$.

- If $x \in X$ belongs to the left side set, then it does not belong to the union $A \cup B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$, so $P(x) \vee Q(x)$ is true. Thus, $\neg(P(x) \vee Q(x))$ gives a boolean formula for membership in the left set.
- If $x \in X$ belongs to the right side set, then it must not belong to A and must not belong to B. Thus, $\neg P(x) \wedge \neg Q(x)$ gives a boolean formula for membership in the right set.

Setting these formulas equal gives $\neg(P(x) \vee Q(x)) \equiv \neg P(x) \wedge \neg Q(x)$, which is an application of the deMorgan's Law $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

(b) Recall that the other deMorgan's law is $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$. Using the ideas from above, translate this into a set equivalence formula, the other "deMorgan's Laws for Sets".

4. Give an example of three sets A, B, and C that satisfy the following properties:

1. $|A \cup C| = 5$
2. $A \cap B \subseteq C$
3. $|B \cap C| = 2$
4. $|B \setminus A| = 4$
5. $|C| = 3$

5. Write out each of the following sets, and their cardinalities.

(a) $A = \mathcal{P}(\emptyset)$: $|A| =$

(b) $B = \mathcal{P}(A)$: $|B| =$

(c) $C = \mathcal{P}(B)$: $|C| =$

(d) $D = A \times C$: $|D| =$

(e) $E = B \times B$: $|E| =$

6.

(a) Given a set S, is it always true that $S \in \mathcal{P}(S)$? If so, explain why. If not, provide a counterexample.

(b) Give an example of a set S such that $S \not\subseteq \mathcal{P}(S)$.

(c) **Challenge:** Give an example of a set S such that $S \subseteq \mathcal{P}(S)$.

7. For each $n \in \mathbb{N}$, define the set $A_n = \{an : a \in \mathbb{N}\}$.

(a) Describe the set A_3.

(b) Describe the set $A_4 \cap A_6$.

(c) **Challenge:** Can you give a description of the set $A_m \cap A_n$ for any $m, n \in \mathbb{N}$? Don't worry about proving your answer now; we'll talk more about this later in the course.