Relations

Review

A **binary relation** on sets S and T is a subset of $S \times T$. We typically think of a relation as encoding some property of pairs (s, t) of elements from these sets.

The **inverse** of a binary relation $R \subseteq A \times B$ is the relation $R^{-1} \subseteq B \times A$ such that

$$(b, a) \in R^{-1} \iff (a, b) \in R.$$

The **composition** of a binary relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is the relation $S \circ R \subseteq A \times C$ such that

$$(a, c) \in S \circ R \iff \exists\, b \in B.\ (a, b) \in R \text{ and } (b, c) \in S.$$

Typically, we take both sets in the Cartesian product defining the relation to be the same set. We call such a relation $R \subseteq S \times S$ a (binary) relation on S.

There are many properties of binary relations that interest us. Suppose that R is a relation on S. Then,

1. R is **reflexive** if $(s, s) \in R$ for each $s \in S$.
2. R is **irreflexive** if $(s, s) \notin R$ for any $s \in S$.
3. R is **symmetric** if whenever $(s, t) \in R$ for some $s, t \in S$, then also $(t, s) \in S$.
4. R is **asymmetric** if whenever $(s, t) \in R$ for some $s, t \in S$, then $(t, s) \notin S$.
5. R is **anti-symmetric** if whenever $(s, t) \in R$ and $(t, s) \in R$, then we must have $s = t$.
6. R is **transitive** if whenever $(s, t) \in R$ and $(t, u) \in R$ for some $s, t, u \in S$, then also $(s, u) \in R$.

A **partial order** is a relation that is reflexive, anti-symmetric, and transitive.

A **strict partial order** is a relation that is irreflexive and transitive.

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

Given an equivalence relation R on S and an element $s \in S$, the **equivalence class** of $[s]$ is the set of all elements $t \in S$ for which $(s, t) \in R$.

The set of all equivalence classes **partition** the set S. Given two equivalence classes $[s], [t]$ (so $s, t \in S$) we either have $[s] = [t]$ (as sets) or $[s] \cap [t] = \emptyset$. This means that every element $s \in S$ belongs to exactly one equivalence class.

The **transitive closure** of a relation R is the smallest transitive relation that contains R as a subset.

1. Consider the possible relations on $S = \{a, b\}$. Each relation consists of a set of pairs of elements from S, and therefore is a subset of $S \times S$. The power set $\mathcal{P}(S \times S)$ contains all possible relations and has size

$$|\mathcal{P}(S \times S)| = 2^{|S \times S|} = 2^{(|S| \cdot |S|)} = 2^{(2 \cdot 2)} = 2^4 = 16.$$

Eight of these relations are given below:

$$R_1 = \{(a,a),(b,b)\} \quad R_2 = \{(a,b),(b,a)\} \quad R_3 = \{(a,a),(a,b),(b,a)\} \quad R_4 = \{(a,a)\}$$
$$R_5 = \{(a,a),(a,b)\} \quad R_6 = \{(b,a),(b,b)\} \quad R_7 = \{(a,b),(b,a),(b,b)\} \quad R_8 = \{(b,b)\}$$

What are the remaining 8 relations?

2. Consider the set $S = \{1, 2, 3\}$. Give an example of a relation on S that is:

(a) Reflexive and Symmetric, but not Transitive.

(b) Reflexive and Transitive, but not Symmetric.

(c) Symmetric and Transitive, but not Reflexive.

3. Consider the relations on $A = \{1, 2, 3, 4\}$ given by:

$$R := \Big\{(1,1),(2,2),(4,4),(2,3),(4,3),(3,1)\Big\}$$
$$S := \Big\{(2,1),(1,2),(3,2),(4,3),(3,4),(3,3),(2,2)\Big\}$$

Describe (as a set of pairs), the following relations.

$$R^{-1} =$$

$$S^{-1} =$$

$$S \circ R =$$

$$R \circ S =$$

$$R^2 =$$

4. Determine whether each of the following relations are:

1. Reflexive
2. Irreflexive
3. Symmetric
4. Anti-symmetric
5. Transitive
6. Asymmetric
7. A Partial Ordering
8. An Equivalence Relation

If the relation is an equivalence relation, describe its equivalence classes.

(a) The greater than relation on the integers, i.e. $a\ R\ b$ when $a > b$.

(b) The inequality relation on the integers, i.e. $a\ R\ b$ when $a \neq b$.

(c) The relation on the integers defined by $a \; R \; b$ when $|a - b| \leq 2$.

(d) The divisibility relation on the integers, i.e. $a \; R \; b$ when a is a divisor of b.

(e) The integers under the relation $a \ R \ b$ when $a - b$ is a multiple of 3.