Predicate Logic

Review

A **predicate** is a statement that includes one or more variables. The truth value of a predicate depends on the values of these variables.

We usually refer to these variables as the **arguments** of the predicate. The **domain** of each argument is the set of possible values that it can take.

There are two logical **quantifiers**, which we can use to form propositions out predicates. Suppose that $P(n)$ is a predicate with argument n ranging over the non-negative integers, $\mathbb{N}$.

1. The existential quantifier ($\exists$) expresses that we can find some value of a variable for which the predicate is true. We read, $\exists\, x \in \mathbb{N}.\, P(x)$ as "There exists x (in $\mathbb{N}$) such that $P(x)$ (is true)."
2. The universal quantifier ($\forall$) expresses that a predicate is true regardless of the value of its argument. We read, $\forall\, x \in \mathbb{N}.\, P(x)$ as "For all x (in $\mathbb{N}$), $P(x)$ (is true)."

To conclude that an existentially-quantified statement $\exists x P(x)$ is true, it suffices to find an **example**, a value in the domain of x for which $P(x)$ is true. To conclude that $\exists x P(x)$ is false, you must show that $P(x)$ is false for every value of x in its domain.

To conclude that a universally-quantified statement $\forall x P(x)$ is true, you must show that $P(x)$ is true for every value of x in its domain. To conclude that $\forall x P(x)$ is false, it suffices to find a **counterexample**, a value in the domain of x for which $P(x)$ is false.

The **DeMorgan's Laws for predicate logic** provide us with rules for negative quantified statements. We have,

$$\neg\Big(\exists\, x.\, P(x)\Big) = \forall\, x.\, \neg P(x)$$
$$\neg\Big(\forall\, x.\, P(x)\Big) = \exists\, x.\, \neg P(x).$$

That is, when we negate a quantified statement we switch the quantifier and negate the predicate inside (where the negated predicate is true for exactly those values of its argument which make the original predicate false).

We can build more complicated propositions by using multiple quantifiers. Consider the following:

$$\forall\, x \in \mathbb{N}\ \exists\, y \in \mathbb{N}.\ y > x.$$

Here "$y > x$" is a predicate involving two arguments, x and y. Then, "$\exists\, y \in \mathbb{N}.\ y > x$" is a predicate involving only the variable x (y is bound to the existential quantifier $\exists$, so the truth value depends only how we fix x). The universal quantifier $\forall$ binds the variable x and makes this statement into a proposition (that is, a statement with a determined truth value).

We read the quantifiers from left to right giving, "For all natural numbers x, there is a natural number y which is greater than x." We can see that this is true by giving the example $y = x + 1$ for each x.

1. Consider the following predicates, each defined over the set of integers ($\mathbb{Z}$):

$$E(x) := x \text{ is even}$$
$$M(x) := x \text{ is a multiple of 3}$$
$$P(x) := x \text{ is prime}$$

In the left column below, there are some statements about a set S. Match these to the logical formulas in the second column. All of the quantifiers in these formulas are over the domain S.

S contains an odd number.	$\forall\, x \in S.\ P(x)$
S contains only prime numbers.	$\neg\exists\, x \in S.\ \Big(P(x) \wedge \neg E(x)\Big)$
S contains a multiple of 6.	$\exists\, x \in S\ \forall\, y \in S.\ \Big(E(x) \wedge (x \leq y)\Big)$
The only prime number in S is 2.	$\exists\, x \in S.\ \neg E(x)$
There are no odd multiples of 3 in S.	$\forall\, x \in S\ \forall\, y \in S.\ E(x - y)$
The smallest number in S is even.	$\exists x \in S.\ \Big(M(x) \wedge E(x)\Big)$
S contains only even numbers or only odd numbers.	$\forall\, x \in S.\ \Big(P(x) \implies (\exists\, y.\ x \leq y)\Big)$
Every number in S is even or prime.	$\forall x \in S.\ \Big(\neg P(x) \implies E(x)\Big)$
The largest number in S is composite.	$\forall\, x \in S.\ \Big(M(x) \implies E(x)\Big)$

2. Let S be the set consisting of all students in a class. Define the following propositions with variables that range over S:

$$\begin{aligned} G(x) &:= x \text{ is a girl.} \\ P(x) &:= x \text{ plays the piano.} \\ F(x,y) &:= x \text{ is friends with } y. \end{aligned}$$

Represent the following sentences using logical symbols.

(a) No one in the class plays the piano.

(b) Every girl in the class plays the piano.

(c) Everyone in the class has a friend in the class.

(d) Someone in the class is friends with everyone.

(e) Everyone in the class either has a friend who is a girl or has a friend who plays the piano (or both).

(f) There is a student in the class who is only friends with girls.

(g) Any two students in the class are either friends or have a friend in common.

3. Determine whether each of the following propositions is true. Explain your answer by providing an example, providing a counterexample, or giving a short explanation. Each of the variables is quantified over the natural numbers ($\mathbb{N}$).

(a) $\forall\, x\, \exists\, y.\ (x + x = y)$.

(b) $\forall\, y\, \exists\, x.\ (x + x = y)$.

(c) $\exists\, y\, \forall\, x.\ (x + x = y)$.

(d) $\exists\, x\, \forall\, y.\ (x + y = y)$.

(e) $\forall\, x\, \forall\, y\, \exists\, z.\ (x + y = z)$.

(f) $\forall\, x\, \forall\, y\, \exists\, z.\ (x + z = y)$.

(g) $\exists\, x\, \exists\, y\, \exists\, z.\ (x^2 + y^2 = z^2)$.

4. For each of the following, determine whether it is a proposition or a predicate. If it is a proposition, determine its truth value. Assume that all quantifiers have domain $\mathbb{Z}$.

(a) $\exists\, x\, \forall\, y.\ (x + y > 0)$

(b) $\exists\, x.\ (2x = y)$

(c) $\exists\, x\, \forall\, y.\ (x^2 + y + x = y)$

5. Explain how it is possible for $\forall\, x \in S.\ P(x)$ to be true, while $\exists\, x \in S.\ P(x)$ is false.