Graphs

Review

A **graph** is a mathematical structure consisting of a set of vertices with edges between them.

An **undirected graph** G is a pair (V, E) where V is a set of **vertices** or **nodes**, and E is a set of **edges** between vertices, 2-sets of the form $\{v, v'\}$ for $v, v' \in V$.

Two vertices $v, v' \in V$ in an undirected graph are **adjacent** if $\{v, v'\} \in V$.

A vertex $v \in V$ and an edge $e \in E$ are **incident** if $v \in E$.

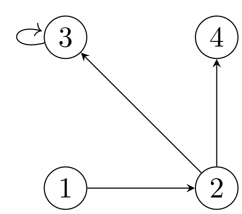
The **degree** of a vertex v in an undirected graph is equal to its number of incident edges.

A **directed graph** G is a pair (V, E) where V is a set of vertices, and $E \subseteq V \times V$ is a set of arcs (directed edges) between vertices.

Given a directed edge $e = (u, v) \in E$, we call the first vertex u the **tail** of the edge and the second vertex v the **head**. We can visualize e as an arrow pointing from u to v.

Given a vertex v in a directed graph, its **in-degree** is the number of edges for which it is the head, and its **out-degree** is the number of edges for which it is the tail.

We can visualize a graph by labeling a circle for each of its vertices and connecting these circles with line segments (for undirected graphs) or arrows (for directed graphs) to visualize the edges. For example, the directed graph G with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$ is visualized on the right. An edge with two of the same vertices is called a **self-loop**.



A **walk** of **length** k is an alternating sequence of $k + 1$ vertices and k edges $u, (u, v), v, (v, w), w, \ldots$ such that each vertex is incident to its adjacent edges in the sequence. Visually, we can trace a walk through a graph in one continuous motion.

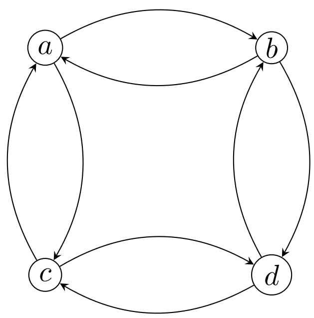
A walk is **closed** if its first and last vertices are the same. Otherwise, it is **open**.

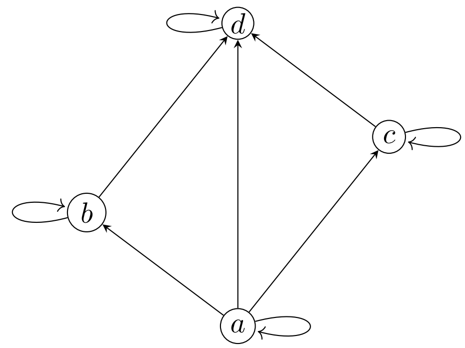
A **path** is an open walk where all of the vertices are distinct.

A **cycle** is closed walk of length at least 1 where all vertices are distinct except for the first and last ones.

1. For each of the following directed graphs, determine whether its edges represent a relation that is:

1. Reflexive
2. Irreflexive
3. Symmetric
4. Anti-symmetric
5. Transitive
6. A Partial Ordering



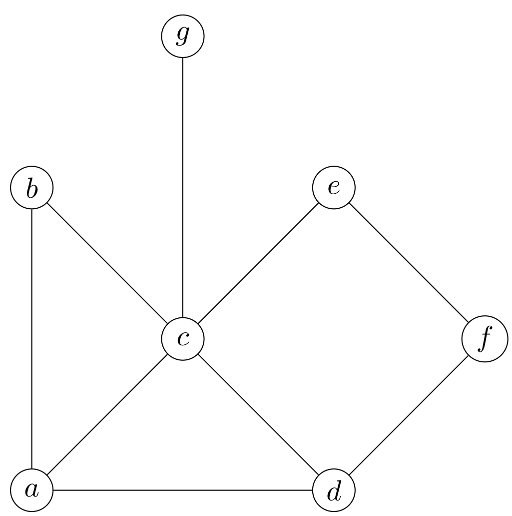


2. Consider the following graph. Given an example of a:

(a) Path of length 6:

(b) Cycle:

(c) Closed walk of length 7:

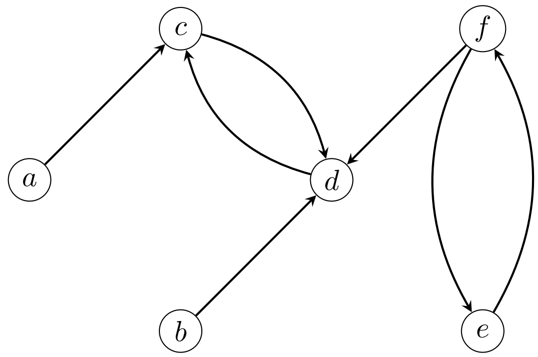


3. In this question, we'll prove some results about the vertex degrees in an undirected graph. Throughout the exercise, you may assume that the graph has no self-loops.

(a) Argue that the sum of degrees of all vertices in an undirected graph is even. (**Hint:** Perform induction on the number of edges in the graph.)

(b) Argue that any undirected graph has an even number of vertices with odd degree. (**Hint:** Use part (a).)

4. Add edges to the following graph to form its transitive closure.



5. Let $G = (V, E)$ be an undirected graph. Define the connection relation R on V such that $v\ R\ w$ if and only if there is a walk from v to w. Argue that R is an equivalence relation on V.

The equivalence classes of R under this connection relation are called the *connected components* of G.