The Algorithmic Landscape of Priority-Respecting Allocations

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Based on joint work with Sid Banerjee and David Kempe
Paper: https://arxiv.org/abs/2204.13019

A Motivating Example: Pandemic Response

Supply-chain constraints place limits on available resources

- Ventilators, Vaccines, Anti-viral treatments

Many considerations for who to prioritize

- Healthcare / essential workers
- Individuals with comorbidities
- Residents of high-density housing

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What is a “fair” way to allocate care?

Commonly used priority schemes have issues

Formalizing the Reserve Allocation Setting

Agents : $\mathcal{A}, \quad n = |\mathcal{A}|$

- *Unit demand* for the resource
- *Indifferent* about how they are allocated

Categories : $\mathcal{C}, \quad m = |\mathcal{C}|$

Each category $c \in \mathcal{C}$ has:

Quota : $q_c \in \mathbb{N}, \quad q = \sum_{c \in \mathcal{C}} q_c$

Eligibility : $\mathcal{E}_c \subseteq \mathcal{A}$

Priorities : *Total pre-order* $\succeq_c$ over $\mathcal{E}_c$

- $\succeq_c$ separates agents into ranked *priority tiers*
- $a \succeq_c a' \implies c$ gives priority to a over a'

Visualizing an Instance

α (2)	β (1)	γ (1)
a	b	b
b	c , e	a
c	d	
d		
e		

Feasible Allocations

Goal: Select an allocation map $\varphi : \mathcal{A} \to \mathcal{C} \cup \{\varnothing\}$

Determines recipient set $\mathcal{A} \setminus \varphi^{-1}(\varnothing)$

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Priority Respecting [PR]: A category allocates to an agent only if all higher-priority agents have been allocated

$$\varphi(a') = c \ \land \ a \succeq_c a' \implies \varphi(a) \neq \varnothing$$

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Locating Good Allocations

Pareto Efficient [PE]: No alternate allocation satisfying [ER], [QR], [PR], allocates to a strict superset of agents

$$\neg\exists\, \psi \;:\; \psi^{-1}(\varnothing) \subsetneq \varphi^{-1}(\varnothing)$$

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Is there an efficient algorithm to find allocations with these properties?

Existing Approaches

Pathak et al (2021) [1]: Variant of Deferred Acceptance [2]

- Agents have arbitrary preferences over eligible categories
- Run DA with agents proposing to categories
- [QR], [ER], [PR], not necessarily [PE]

Delacrétaz (2021) [3]: Simultaneous Reserves Algorithm

- "Water-filling" down priority lists determines who gets allocated
- [QR], [ER], [PR], not necessarily [PE]

Aziz and Brandl (2021) [4]: Reverse Rejecting Algorithm

- Iteratively certifies whether a maximal allocation can be found without allocating to a particular agent
- All four properties, but requires $O(n)$ max matching problems

Toward an Efficient Algorithm

Decision variables: $\mathbf{x} = \{x_{a,c}\}_{a \in \mathcal{A}, c \in \mathcal{C}}$. $x_{a,c} = \mathbb{I}\big(\varphi(a) = c\big)$,

(P_0)

$$
\begin{aligned}
\max \quad & V(\mathbf{x}) := \sum_{a \in \mathcal{A}} \sum_{c \in \mathcal{C}} x_{a,c} & & \text{[PE]} \\
\text{s.t.} \quad & \sum_{a \in \mathcal{A}} x_{a,c} \leq q_c & \forall\, c \in \mathcal{C} & \quad \text{[QR]} \\
& \sum_{c \in \mathcal{C}} x_{a,c} \leq 1 & \forall\, a \in \mathcal{A} & \quad \text{[UD]} \\
& x_{a,c} = 0 & \forall\, a, c \;:\; a \notin \mathcal{E}_c & \quad \text{[ER]} \\
& x_{a,c} \in \{0, 1\} & \forall\, a \in \mathcal{A}, c \in \mathcal{C} &
\end{aligned}
$$

(P_0) encodes a bipartite b-matching problem

LP-relaxation is totally unimodular $\implies$ integer corner points

(P_0) Doesn't Account for Priorities

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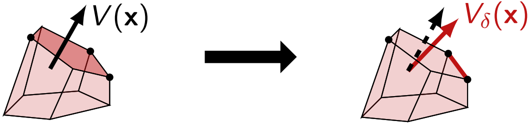
To incorporate priorities, we'll modify the IP objective.

Adding Priorities

Idea: Tilt the objective so remaining optima respect priorities



Replace $V(\mathbf{x})$ with $V_\delta(\mathbf{x}) = \sum_{a\in\mathcal{A}}\sum_{c\in\mathcal{C}}\left(1-\delta_{a,c}\right)x_{a,c}$.

Interpreting $\delta_{a,c}$ as the cost of allocating *a* through *c*, a *valid* δ satisfies:

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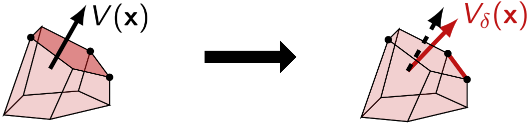

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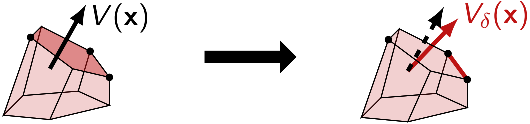

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Consistent: Prioritized agents have lower cost $a \succeq_c a' \iff \delta_{a,c} \leq \delta_{a',c}$

Our Perturbed LP

Given any δ, define the LP

(P_δ)

$$\begin{aligned}
\max \quad & V_\delta(\mathbf{x}) & \\
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Theorem

Let $\mathbf{x}^*$ be a solution of (P_δ) for any valid δ. Then, $\mathbf{x}^*$ corresponds to an allocation satisfying $[ER], [QR], [PR], [PE]$.

Realizability of Good Allocations

Converse result:

Theorem (Informal)

Every recipient set determined by a fair allocation can be located by solving (P_δ) for some valid δ.

- Our perturbed matching framework is a standard setting
- The restrictions we've placed on δ are minimal

How far can we extend our techniques to handle related problems?

3 Case Studies: "Computational knife's edge" of priority-respecting allocation

1. Reasoning about a Particular Agent

Must agent *a* be allocated?

Remove *a* and all lower-ranked agents from instance.

Check if max matching size decreases

α (2)	β (1)	γ (1)
w	*x*	*x*
x	*a* , *z*	*w*
a	*y*	
y		

Can agent *a* be allocated?

A *serviceable* agent is a recipient in *some* good allocation.

Deciding whether an agent *a* is serviceable is NP-Hard.

Proof Idea: Reduction from X3C problem.

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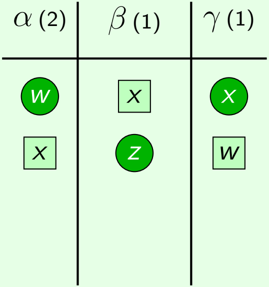

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2. Incorporating Agent Utility

Agent a has a utility function $u_a : \mathcal{C} \to (0, 1]$ encoding preference for certain categories. ($u_a(\varnothing) = 0$.)

Utility *Pareto-Efficient* Allocation

Run our algorithm twice

First run uses arbitrary δ to determine recipient set.

Second run removes agents outside of recipient set and sets δ according to agent utilities.

Utility *Maximizing* Allocation

NP-Hard via a reduction from serviceable problem.

Proof Idea: One agent has high utility in all categories, others have low utility.

*Hardness reduction can be generalized to other optimization objectives (e.g. Nash Social Welfare)

3. Global Fairness Heuristics

For each eligible agent $a \in \mathcal{E}_c$, let $r_c(a)$ be their priority tier in c ($1 =$ highest priority, $2 =$ next priority tier, etc.)

Minimizing Maximum Allocated Rank

Run algorithm with "geometric" perturbation

$$\delta_{a,c} \propto n^{r_c(a)}$$

Proof Idea: Cost of highest ranked allocation dominates all others.

Maximizing Minimum Unallocated Rank

NP-Hard via an XC3 reduction similar to serviceable problem.

Proof Idea: Serviceable candidate has low rank in their only category. Other categories fill more ranks.

Conclusion

- Reserve Allocation is a reasonable modeling framework for assignment problems with “competing” objectives
- Can locate good allocations via a weighted matching LP
  - More efficient than existing approaches
  - Provides flexibility to many problem extensions

Open Questions:

Natural desiderata that locate a unique (fractional) allocation?

Can this allocation be computed efficiently?

- Our perturbation technique seems useful in other related problems

Thank You!

References

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Proving the Main Theorem

Theorem

Let $\mathbf{x}^*$ be a solution of (P_δ) for any valid δ. Then, $\mathbf{x}^*$ corresponds to an allocation satisfying $[ER], [QR], [PR], [PE]$.

Proof Sketch.

[ER],[QR]: Ensured by (P_δ) constraints.

[PR]: δ is Consistent.

[PE]: Small Effect of δ and integrality:

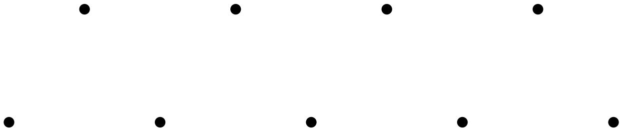
$$V(\hat{\mathbf{x}}) \geq V(\mathbf{x}^*) \geq V_\delta(\mathbf{x}^*) \geq V_\delta(\hat{\mathbf{x}}) = V(\hat{\mathbf{x}}) - \sum_{a,c} \delta_{a,c} \geq V(\hat{\mathbf{x}}) - \tfrac{1}{2}.$$

so $V(\hat{\mathbf{x}}) = V(\mathbf{x}^*)$ for a solution $\hat{\mathbf{x}}$ to (P_0). □

Exact Cover by 3-Sets (X3C)

Input: Ground set $E = \{e_1, e_2, \ldots, e_{3n}\}$.
Collection of subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$, each $|S_i| = 3$.

Decide: Is there a collection of subsets $\{S_{i_1}, \ldots, S_{i_n}\}$ such that $E = \bigcup_{j=1}^{n} S_{i_j}$?



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X3C *is NP-Complete.*

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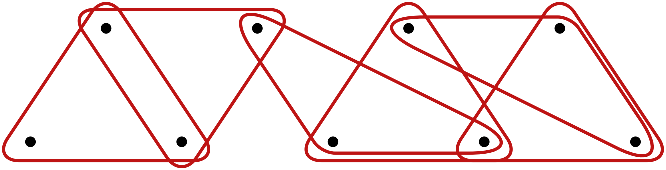

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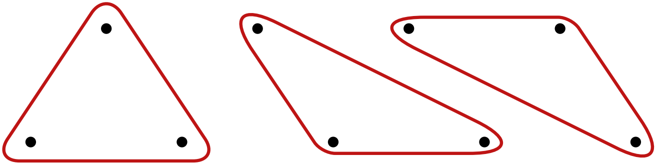

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The Reduction

X3C Input: $E = \{e_1, e_2, \ldots, e_{3n}\}$, $\mathcal{S} = \{S_1, \ldots, S_m\}$

$S_j = \{e_{i_{j,1}}, e_{i_{j,2}}, e_{i_{j,3}}\}$

Allocation Instance: $\mathcal{A} = E \cup \{s_1, \ldots, s_m\} \cup \left\{f_1, \ldots, f_{4(m-n)}\right\} \cup a$

set categories

α_1 (4)	$\cdots$	α_m (4)	β (1)
f_1		f_1	e_1
$\vdots$		$\vdots$	$\vdots$
$f_{4(m-n)}$		$f_{4(m-n)}$	e_{3n}
s_1		s_m	a
$e_{i_{1,1}}$		$e_{i_{m,1}}$	
$e_{i_{1,2}}$		$e_{i_{m,2}}$	
$e_{i_{1,3}}$		$e_{i_{m,3}}$	